# 機械力学４回目演習回答

１．

（１）

質量$m_1$のブロックの位置$(x_1, 0)$

質量$m_2$の球の位置$(x_1 + r\sin\theta, -r\cos\theta)$

質量$m_1$のブロックの速度$(\dot{x}_1, 0)$

質量$m_2$の球の速度$(\dot{x}_1 + r\dot{\theta}\cos\theta, r\dot{\theta}\sin\theta)$

運動エネルギー$T$は,

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\left\{(\dot{x}_1 + r\dot{\theta}\cos\theta)^2 + r^2\dot{\theta}^2\sin^2\theta\right\} \\ &= \frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2(\dot{x}_1{}^2 + 2r\dot{\theta}\dot{x}_1\cos\theta + r^2\dot{\theta}^2) \end{aligned} \tag{１}$$

（２）ポテンシャルエネルギーは$U$は

$$U = -m_2 gr\cos\theta \tag{２}$$

ラグランジュ関数は，式（１），（２）より，

$$L = T - U = \frac{m_1 + m_2}{2}\dot{x}_1{}^2 + m_2 r\dot{\theta}\dot{x}_1\cos\theta + \frac{m_2}{2}r^2\dot{\theta}^2 + m_2 gr\cos\theta \tag{３}$$

$\theta$が十分に小さいより， $\cos\theta \fallingdotseq 1, \sin\theta \fallingdotseq \theta, \dot{\theta}^2 \fallingdotseq 0$とすると，

$x_1$に関して，

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + m_2 r\dot{\theta}\cos\theta \fallingdotseq (m_1 + m_2)\dot{x}_1 + m_2 r\dot{\theta} \tag{４}$$

$$\frac{\partial L}{\partial x_1} = 0 \tag{５}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = (m_1 + m_2)\ddot{x}_1 + m_2 r\ddot{\theta} = 0 \tag{６}$$

$\theta$に関して，

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 r\dot{x}_1\cos\theta + m_2 r^2\dot{\theta} \fallingdotseq m_2 r\dot{x}_1 + m_2 r^2\dot{\theta} \tag{７}$$

$$\frac{\partial L}{\partial \theta} = -m_2 r\dot{\theta}^2\dot{x}_1\sin\theta - m_2 gr\sin\theta \fallingdotseq -m_2 gr\theta \tag{８}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=m_2 r\ddot{x}_1+m_2 r^2\ddot{\theta}+m_2 gr\theta=0 \qquad (9)$$

２．

運動エネルギー$T$は,

$$T=\frac{1}{2}m_1(\dot{y}+\dot{y}_1)^2+\frac{1}{2}m_2(\dot{y}+\dot{y}_2)^2+\frac{1}{2}M\dot{y}^2+\frac{1}{2}I(\frac{\dot{y}_1}{R})^2 \qquad (1)$$

ここで，$\dot{y}_1=-\dot{y}_2$（２）なので，式（１）にこの関係を代入すると,

$$T=\frac{1}{2}m_1(\dot{y}+\dot{y}_1)^2+\frac{1}{2}m_2(\dot{y}-\dot{y}_1)^2+\frac{1}{2}M\dot{y}^2+\frac{1}{2}I(\frac{\dot{y}_1}{R})^2 \qquad (3)$$

ポテンシャルエネルギー$U$は,

$$U=\frac{1}{2}k(y-y_0)^2-m_1 g(y+y_1)-m_2 g(y+y_2)-Mgy \qquad (4)$$

ここで，式（２）より，$y_1=\alpha-y_2$，$\alpha$は定数なので，式（４）にこの関係を代入すると,

$$U=\frac{1}{2}k(y-y_0)^2-m_1 g(y+y_1)-m_2 g(y+\alpha-y_1)-Mgy \qquad (5)$$

ラグランジュ関数$L$とすると,

$$\begin{aligned}L=T-U&=\frac{1}{2}m_1(\dot{y}+\dot{y}_1)^2+\frac{1}{2}m_2(\dot{y}-\dot{y}_1)^2+\frac{1}{2}M\dot{y}^2+\frac{1}{2}I(\frac{\dot{y}_1}{R})^2\\&-\frac{1}{2}k(y-y_0)^2+m_1 g(y+y_1)+m_2 g(y+\alpha-y_1)+Mgy\end{aligned} \qquad (6)$$

$y_1$に関して

$$\frac{\partial L}{\partial \dot{y}_1}=m_1(\dot{y}+\dot{y}_1)-m_2(\dot{y}-\dot{y}_1)+\frac{I}{R^2}\dot{y}_1 \qquad (7)$$

$$\frac{\partial L}{\partial y_1}=m_1 g+m_2 g \qquad (8)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}_1}\right)-\left(\frac{\partial L}{\partial y_1}\right)=m_1(\ddot{y}+\ddot{y}_1)-m_2(\ddot{y}-\ddot{y}_1)+\frac{I}{R^2}\ddot{y}_1-m_1 g+m_2 g=0 \qquad (9)$$

$y$に関して

$$\frac{\partial L}{\partial \dot{y}}=m_1(\dot{y}+\dot{y}_1)+m_2(\dot{y}-\dot{y}_1)+M\dot{y} \qquad (10)$$

$$\frac{\partial L}{\partial y} = -k(y - y_0) + (m_1 + m_2 + M)g \tag{１１}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \left(\frac{\partial L}{\partial y}\right)\\ &= (m_1 + m_2 + M)\ddot{y} + (m_1 - m_2)\ddot{y}_1 + k(y - y_0) - (m_1 + m_2 + M)g = 0\end{aligned} \tag{１２}$$

式（９）を変形すると，

$$\ddot{y}_1 = \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}(g - \ddot{y}) \tag{１３}$$

式（１３）を式（１２）に代入すると，

$$\ddot{y} = \frac{-(m_1 + m_2 + \frac{I}{R^2})k}{(m_1 + m_2 + M)(m_1 + m_2 + \frac{I}{R^2}) - (m_1 - m_2)^2}(y - y_0) + g \tag{１４}$$

３．

運動エネルギー$T$について，

質量$M_1$の台車は，

$$T_{M_1} = \frac{1}{2}M_1\dot{x}^2 \tag{１}$$

質量$M$の車輪（４個）は，

$$T_M = \left\{\frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2\right\} \times 4 \tag{２}$$

質量$m$のブロックは，

$$T_m = \frac{1}{2}m\left(\dot{x} + \frac{\dot{y}}{\tan\theta}\right)^2 + \frac{1}{2}m\dot{y}^2 \tag{３}$$

よって

$$\begin{aligned}T &= T_{M_1} + T_M + T_m\\ &= \frac{1}{2}\left(M_1 + 4M + m + 4\frac{I}{r^2}\right)\dot{x}^2 + m\frac{\dot{x}\dot{y}}{\tan\theta} + \frac{m}{2}\frac{1}{\sin^2\theta}\dot{y}^2\end{aligned} \tag{４}$$

ポテンシャルエネルギー$U$について，

$$U = mgy + \frac{1}{2}k\left(q - q_0\right)^2 = mgy + \frac{1}{2}k\left(\frac{y_0 - y}{\sin\theta} - q_0\right)^2 \qquad （５）$$

ラグランジュ関数$L$は

$$L = T - U$$

$$= \frac{1}{2}\left(M_1 + 4M + m + 4\frac{I}{r^2}\right)\dot{x}^2 + m\frac{\dot{x}\dot{y}}{\tan\theta} + \frac{m}{2}\frac{1}{\sin^2\theta}\dot{y}^2 - mgy - \frac{1}{2}k\left(\frac{y_0 - y}{\sin\theta} - q_0\right)^2$$

（６）

$x$について，

$$\frac{\partial L}{\partial \dot{x}} = \left(M_1 + 4M + m + \frac{4I}{r^2}\right)\dot{x} + \frac{m}{\tan\theta}\dot{y} \qquad （７）$$

$$\frac{\partial L}{\partial x} = 0 \qquad （８）$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \left(M_1 + 4M + m + \frac{4I}{r^2}\right)\ddot{x} + \frac{m}{\tan\theta}\ddot{y} = f \qquad （９）$$

$y$について，

$$\frac{\partial L}{\partial \dot{y}} = \frac{m\dot{x}}{\tan\theta} + \frac{m}{\sin^2\theta}\dot{y} \qquad （１０）$$

$$\frac{\partial L}{\partial y} = -mg + \frac{k}{\sin\theta}\left(\frac{y_0 - y}{\sin\theta} - q_0\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{m\ddot{x}}{\tan\theta} + \frac{m}{\sin^2\theta}\ddot{y} + mg - \frac{k}{\sin\theta}\left(\frac{y_0 - y}{\sin\theta} - q_0\right) = 0 \qquad （１１）$$

式（１１）を整理すると

$$\frac{m\ddot{x}}{\tan\theta} + \frac{m}{\sin^2\theta}\ddot{y} = -mg + \frac{k}{\sin\theta}\left(\frac{y_0 - y}{\sin\theta} - q_0\right) \qquad （１２）$$